Structures and universality: Rough solutions in geometric analysis and the calculus of variations.

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Energy driven pattern formation

Cooled from above

Heated from below

\[ E = \int W(Du) + \epsilon^2 \int F(D^2 u) \]

\( W \) is non convex, \( F \) is a convex regularizer.
Part I
Patterns and the Cross-Newell energy
convection rolls and pattern formation

Pr = 1.4

courtesy of Eberhard Bodenschatz

A variational formulation

\[ \tau(k^2) \frac{\partial \Theta}{\partial \Gamma} = -\nabla \cdot (k B(k^2) + \varepsilon^2 \Delta k) = -\frac{\delta \mathcal{F}}{\delta \Theta} \]

for an appropriate energy functional \( \mathcal{F} \).

For \( k \sim k_c = 1 \),

\[ \tau(k^2) \approx 1, \quad B(k^2) = 2(1 - k^2), \]

and

\[ \mathcal{F} = \frac{1}{2} \int \left[ \varepsilon^2 (\Delta \Theta)^2 + (1 - |\nabla \Theta|^2)^2 \right] dX dY \]

Aviles-Giga energy functional, thin-film blistering.

As \( \varepsilon \to 0 \), the expectation is that \( |\nabla \Theta| = 1 \) a.e. – the Eikonal equation.

Singularities are one-dimensional.

- Ambrosio, DeLellis and Mantegazza, DeSimone, Kohn, Müller and Otto, Ercolani and Taylor.
Line vs. point defects

Eikonal equation

Swift-Hohenberg numerics
Concentration of Energy

Swift-Hohenberg equation in an ellipse
Rob Indik et al
Convex and Concave Disclinations


Courtesy of G. Ahlers, U.C. Santa Barbara
http://tweedledee.ucsb.edu/~guenter/picturepage5.html
Twist

Concave disclination
A Dislocation

Image Courtesy J. Lega
The phase $\theta$

$F(\theta) \sim \cos(\theta)$

Periodicity and symmetry imply we have to identify $\theta$ with $\pm \theta + 2n\pi$.

- The values $\theta = n\pi$ are distinguished – at these values $\theta = -\theta + 2n\pi$ are identified

- *continuous symmetry* $\theta \rightarrow \theta + \delta$ is broken.

Square: Smooth function $\theta_1(x, y)$

Circle: Smooth function $\theta_2(x, y)$

Overlap: $\exists \epsilon \in \{1, -1\}$

$k \in \mathbb{Z}$

such that $\theta_2 = \epsilon \theta_1 + 2k\pi$
The “phase” space

“Tangent” bundle of the phase space

\[ \text{“} TM \text{”} = (0, \pi) \times \mathbb{R} \bigcup \{0, \pi\} \times \mathbb{R}/\{-1, 1\} \]
Vectors vs. directors

up-down symmetry

Symmetry allows more "freedom" for directors
Defect solutions and effective energies

Basic field is $k = \nabla \theta$.

Effective energy:

Analogy with Mumford-Shah functional. Sum of bulk part, line energy and point defect energy.

Quantization: $\nabla \times k = \sum_i \pi d_i \delta_{z_i}$

Loop defects
Part II
Hyperbolic elastic sheets
Morphology: growth and deformation
Negatively curved elastic sheets

Multiple scale buckling

“Buckling cascade in free thin sheets”, E. Sharon, et al.
Geometry: The Gauss Normal map

\[ N(p) \cdot dr(p) = 0 \]

\[ I \equiv ds^2 = dr(p) \cdot dr(p) \]

\[ II = -dr(p) \cdot dN(p) = d^2r(p) \cdot N(p) \]
Elastic energy of a thin sheet

\[ \gamma = (D\Phi)^T \cdot D\Phi - g \]
\[ \kappa = \hat{n} \cdot D^2\Phi \]

\[ E = \int \|\gamma\|^2 + \epsilon^2 \|\kappa\|^2 \]

Stretching energy  Bending energy

Small slopes:  \( u = x + h^2 \xi(x, y), v = y + h^2 \zeta(x, y), w = h\eta(x, y) \)
Elastic energy

Reference Riemannian metric $g$. 

Immersion $\Phi : \Omega \to \mathbb{R}^3$ of the center surface.

$$\gamma = D\Phi^T \cdot D\Phi - g$$

$$E^t[\Phi] = S[\gamma] + t^2 B[H, K]$$

$$= \int_\Omega Q(\gamma) \, dx \, dy + t^2 \int_\Omega (4H^2 - 2K) \, dx \, dy,$$

Lewicka and Pakzad (2011). $\Gamma$–limit:

$$\lim_{t \to 0} t^{-2} E^t = \begin{cases} 
\int_\Omega (4H^2 - 2K) \, dx \, dy & \text{if } \Phi \in W^{2,2}_{iso} \\
+\infty & \text{otherwise}
\end{cases}$$
Hilbert’s Theorem

- Hilbert (1901): In $\mathbb{E}^3$ there is no complete, analytic surface of constant negative curvature.
- Efimov (1968): In $\mathbb{E}^3$ there is no complete, $C^2$ surface with curvature $K \leq c < 0$.
- Nash (1954): In $\mathbb{E}^3$ there exist complete, $C^1$ surfaces with $K \leq c < 0$

“Theorem” (A Quantitative Hilbert’s Theorem (TS, SV))

For the elastic energy given previously,

\begin{align*}
  \text{i) } & \inf_{C^{1,1}} \mathcal{E} \lesssim \exp R^{1/2} \\
  \text{ii) } & \inf_{C^2} \mathcal{E} \gtrsim \exp R
\end{align*}
What is a $C^{1,1}$ solution?

Small slopes approximation: \( \det(\nabla\nabla w) = -1 \)

Solutions: \( w = \frac{1}{2} \left( ax^2 - \frac{y^2}{a} \right) \).

\( w = 0 \) for \( y = \pm ax \). Pick \( a = \cot(\pi/n) \).
Index of a branch point

\[ C^2 \text{ isometries are not dense in } W^{2,2} \text{ isometries!} \]

Energy gap?
Consequences for Numerics?
Geodesics and Asymptotic lines: $C^2$ surfaces


Normal curvature is zero.  Tangential curvature is zero.
Piecwise quadratic surfaces

Let us consider solutions of \( \det(D^2w) = -1 \)

\[
\begin{align*}
    w(x, y) &= \begin{cases} 
        xy - y^2 \cot(\theta_+) & 0 \leq \theta \leq \theta_+ \\
        xy + y^2 \cot(\theta_-) & -\theta_- \leq \theta \leq 0
    \end{cases}
\end{align*}
\]

\( w \) is \( C^{1,1} \).

All the straight lines through any point lie in a common plane.

Construction of $C^{1,1}$ Isometries

Lelieuvre Formulae:

- Using these connections between $r$ and $N$, restricting ourselves to $K = -1$ surfaces:

  $$N_{uv} \times N = 0$$

- Equivalently

  $$(N \times N_u)_v + (N \times N_v)_u = 0$$

  and

  $$r_u = N_u \times N, \quad r_v = -N_v \times N$$

With prescribed boundary data for $N$ along $\{u = 0\}$ and $\{v = 0\}$, satisfying Beltrami-Enneper.
Construction of $C^{1,1}$ Isometries
Distributional Sine-Gordon equation

\[ \omega = \phi_v dv - \phi_u du \]
\[ d\omega = \phi_{uv} du \wedge dv \]

Intrinsic form of Sine-Gordon:

\[ d\omega = \sin(\phi) du \wedge dv \]

On an appropriate covering space:

\[ \phi_{uv} = \sin(\phi) - 2\pi \sum_i d_i \delta_{z_i} \]
Optimal control for Coupled Pendulums

Minimize $\|\phi - \frac{\pi}{2}\|_\infty$ on the unit square $[0, 1]^2$ over solutions of $\phi_{uv} = \lambda^2 \sin \phi$.

For a simple pendulum, the minimum scales

$$\frac{\pi}{2} - C \log(\lambda)$$
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