Geometry and mechanics of non-Euclidean thin sheets.

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Morphology: growth and plastic deformation

Images courtesy Eran Sharon and John Gemmer
Negatively curved elastic sheets

Multiple scale buckling

“Buckling cascade in free thin sheets”, E. Sharon, et al.
The observed curvature of the arcs when they are flattened is called the geodesic curvature along these lines—another property controlled by their metric. An important observation is that the geodesic curvature along the edges of the wavy leaf in Figure 8 is nearly constant. We do not see any big variations in this curvature that are correlated either with the vein structure or with the waviness of the leaf. The tissue along the edge grew nearly uniformly, the growth law was uniform, and the leaf grew as a simple leaf. Like the plastic sheets, it should have been a simple featureless leaf, but because of the geometrical limitations of space, it was forced to break the symmetry and to adopt a wavy shape.

Wrapping Up

Flowers, like leaves, form complex buckled shapes. Geometrically, the main difference between the two is that leaves form essentially from long, free-standing strips, whereas flowers have more complex geometries; the central tube of a daffodil, for example, closes on itself like a cylinder. What happens to such a cylinder or tube when we apply to it a metric that increases toward its edge? Just as the leaf grows from the center, we can think about “growing” such cylinders starting from a ring of cells and adding rings on top of one another. If the rings all have the same number of cells, they will have the same diameter and will form a cylinder. However, as the number of cells that form a ring grows exponentially upward, the metric of the cylinder increases also, leading to an increasing diameter of the cylinder in its upper part and to a trumpet-like shape. As the metric of the flower increases, the edge of the flower splays outward more and more. Eventually, it splays out so much that the edge of the flower is perpendicular to the direction of the stem along which it is growing. It forms a circle with a radius we’ll call $R$. That marks the end of this phase of flower growth. If cells continue to attach to the end of the flower, causing the metric to grow at an ever-steeper rate as the flower grows sideways, the perimeter of the edge will have to be longer than $2\pi R$. This is known to be impossible in our Euclidean space without breaking the axial symmetry. The edge of the flower must buckle. In Figure 9a we show the result of an experimental study using thin tubes made of polyacrylamide gel. This gel changes its volume depending on its environment. It swells in water, but shrinks in acetone. We used this property to change the metric of the tube. First, we dipped the tube in acetone, causing it to shrink uniformly. Next we dipped one end of the tube in water, allowing the water to diffuse into the tube. As a result, the tube buckled.
Experimental observations: Refinement with decreasing thickness

Experiments by Yael Klein
Halftone Gel Lithography

1. Resist dots embedded in polymer
2. All dots same size
3. Dot size changes
4. Swelling occurs

3D Micro-printing

With a proper half tone pattern of resist dots, almost any 3D shape can be achieved.

Polymer height exaggerated

Buckling occurs from the mismatch in growth from one location on the sheet to another.

About 600 μm (approximately the width of a mechanical pencil lead).

Image credit: Zina Deretsky, National Science Foundation via Chris Santangelo
Elastic energy of a thin sheet

\[ E = \int \| \gamma \|^2 + \varepsilon^2 \| \kappa \|^2 \]

\[ \gamma = (D\Phi)^T \cdot D\Phi - g \]

\[ \kappa = \hat{n} \cdot D^2\Phi \]

\[ \mathcal{E} = \int \| \gamma \|^2 + \varepsilon^2 \| \kappa \|^2 \]

Stretching energy

Bending energy

Small slopes: \( u = x + h^2 \xi(x, y) \), \( v = y + h^2 \zeta(x, y) \), \( w = h\eta(x, y) \)
Modeling Assumptions

• The observed configuration is a minimizer for the elastic energy.

\[ \mathcal{E} = \int \|\gamma\|^2 + \epsilon^2 \|\kappa\|^2 \]

• If there exist smooth configurations with \( \gamma = 0 \) (Isometric immersions), then the configuration of the elastic sheet should converge to this limit as the thickness is reduced.

• Conversely, if the Bending content \( B = \int \|\kappa\|^2 \) diverges as \( \epsilon \to 0 \), there do not exist smooth isometric immersions for the given metric \( g \).

• Experimental sheets are sufficiently thin that they are described by the \( \epsilon \to 0 \) asymptotics.

• The small slopes (Föppl von Kármán) approximation adequately captures all the important features of the system.

• The dynamics of the system does not play a significant role.
Experimental observations: Refinement with decreasing thickness

Experiments by Yael Klein
Geometry: The Gauss Normal map

\[ \mathbf{N}(p) \cdot d\mathbf{r}(p) = 0 \]

\[ I \equiv ds^2 = d\mathbf{r}(p) \cdot d\mathbf{r}(p) \]

\[ II = -d\mathbf{r}(p) \cdot d\mathbf{N}(p) = d^2\mathbf{r}(p) \cdot \mathbf{N}(p) \]
Geometry of surfaces in one slide:

The first fundamental form $I$ determines the (symmetric) metric tensor $g$

The second fundamental form $II$ determines the (symmetric) curvature tensor $\kappa$

Gauss Curvature: $K = \det(g^{-1} \kappa)$

Gauss’ Theorema Egregium: $g$ determines $K$.

Hyperbolic surface: $K < 0$.

Flat Sheet: $K = 0$
Hyperbolic Monge Ampere equations

\[ N = \frac{n}{\|n\|}. \]

\[ d\omega = \frac{dA}{\|n\|^4} \]

\[ dp \wedge dq = \frac{K(x,y)}{\|n\|^4} dx \wedge dy \]

When \( K < 0 \), these are Hyperbolic Monge-Ampere equations.

For any domain on which the normal map is one-to-one, the area of the spherical image cannot exceed \( 2\pi \). No such restriction for the planar image.

**Conjecture:** “Lifting principle”.

\[ n = (-w_x, -w_y, 1) \]

\[ dA = (w_{xx}w_{yy} - w_{xy}^2) dx \wedge dy. \]
Geodesics and Asymptotic lines: $C^2$ surfaces

Tangential curvature is zero. Normal curvature is zero.

Negatively curved sheets: Disk geometry

Small slopes approximation: \[ \det(\nabla \nabla w) = -1 \]

Solutions: \[ w = \frac{1}{2} \left( ax^2 - \frac{y^2}{a} \right). \]

\[ w = 0 \text{ for } y = \pm ax. \] Pick \( a = \cot(\pi/n). \)

**Theorem:** (J. Gemmer, SV). \( D \) is the unit disk with a metric whose FvK curvature is \(-1\). For all \( n \in \mathbb{N} \), we have a \( n \)-periodic local minimizer for the elastic energy, whose energy satisfies the bounds

\[
\min(C_1, C_2 nt^2) \leq E_{FvK} \leq \min(c_1, c_2 n^2 t^2).
\]
Energy of the patched saddle surfaces

No refinement with thickness!
Hyperbolic disks: Full geometry

By allowing non-smooth embeddings, we can decrease the curvature, and the energy.

It is often enough to allow points where the embedding is not $C^2$, but is still $C^{1,1}$. Pogorelov, Zalgaller, Shikin, Poznyak,...
Crochet Images courtesy of Gabriele Meyer.
Branch points: non-$C^2$ surfaces

(a) Three subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
(b) Nine subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
(c) The nine subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
(d) The nine subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
(e) The nine subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
(f) The nine subwrinkle solution created by inserting nine copies of the solution of the unit disk with an intermediate branch point.
Piecewise smooth multi-scale buckled solution
Self-similar branching

Branch points

Curvature determined by iteration number
Ongoing work

- Understand the role of these non-$C^2$ solutions in physically observed patterns
- Rigidity/flexibility in solutions of nonlinear hyperbolic equations
- Discretize this construction.
- Inverse problems
References:


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