Bessel beams as conical waves

The expression
\[ J_n(\alpha r) \cos(n \beta z) \]
describes a superposition of plane waves whose wave vectors all lie on the surface of a cone → No diffraction. The ideal Bessel beam \( A_0 = 1 \) has infinite energy.

Generating finite energy conical waves with an axicon

Apodization: A Gaussian beam propagating in positive \( z \)-directions through an axicon generate a finite energy approximation to a Bessel beam.

Bessel wave equation

Statement of the Problem: Given a solution \( u(x,y,z) \) of the paraxial wave equation (PWE)

\[ \frac{\partial^2 u}{\partial y^2} + n^2 k^2 \frac{\partial^2 u}{\partial z^2} - \left(n^2 k^2 + g \right) \frac{\partial u}{\partial z} - j \omega \frac{\partial u}{\partial t} = 0 \]

can we describe analytically the effects of a Gaussian apodization of this solution?

Propagator of an axicon

Transverse profile is the smooth order Bessel function.

Idea: Apodization of Bessel beams

Theorem: Let \( u(x,y) \) be the solution of the PWE with initial conditions \( u(x,0) = a(x) \). Define for \( p \in \mathbb{C} \) and \( j = 1, \ldots, 2n \) the coordinate transforms

\[ \begin{align*}
    x &= e^{-j \theta} x' + e^{j \theta} x'' \\
    y &= y' + j (y'' - \theta) \\
    z &= z' + j (z'' + \theta)
\end{align*} \]

Then the solution of the PWE with

\[ u(x,y,z) = \exp(\exp(a(x) / z')) \]

is given by

\[ u(x,y,z) = \exp(\exp(a(x) / z')) \]

Ideal Bessel pulse

For the ideal (infinite energy) pulse, the spectrum is concentrated along the line

\[ k_x = \left( k_0 - (Z - \chi) \right)/2 \]

Apodized pulse

With apodization, the support of the spectrum "thickens" and is a narrow strip along the curve.

Pulse propagation

On axis field for a finite energy pulse which is spatially and temporally localized.

Nonlinear conical waves: Apodization + Nonlinear Self-Interaction

In 1-2D we will consider a nonlinear modification of PWE

\[ j \omega \frac{\partial u}{\partial t} + \left( \frac{\partial^2 u}{\partial y^2} + n^2 k^2 \frac{\partial^2 u}{\partial z^2} - \left(n^2 k^2 + g \right) \frac{\partial u}{\partial z} \right) - j \omega \frac{\partial u}{\partial t} + \alpha |u|^2 u = 0 \]

This function \( A_0 \) describes a spatiotemporal instability. For example

\[ A_0(x,y) \approx \exp(\exp(-x^2)) \]

Nonlinear self-interaction

We have to solve the nonlinear initial conditions to obtain a finite energy solution.

\[ \begin{align*}
    u(x,y,0) &= \exp(\exp(-x^2)) \\
    \frac{\partial u}{\partial z}(x,y,0) &= 0
\end{align*} \]

Solutions

We assume that \( n^2 k^2 \ll \chi \ll \omega \). Then for \( \alpha > 0 \) we obtain that

\[ u(x,y) = A_0(x,y) \exp(\exp(-x^2)) \]

Asymptotic solutions

We obtain, for example, an asymptotic solution of the form

\[ u(x,y) = \exp(\exp(-x^2)) \]

Key result at this order is a Nonlinear Phase shift